tions, and hence less than the inferred great thickness of many atoll masses, but it would presumably be sufficient to cause a moderate preponderance of submergence on continental coasts which themselves suffer many diverse movements of upheaval and depression. It is not, however, to be supposed that general warpings and deformations of the ocean floor, upward and downward, should be left out of consideration; such movements have surely taken place to a less or greater degree, particularly in the western Pacific, where coral reefs border continental islands. The integrated effect of all these causes of change in the level of the ocean surface cannot now be determined, because so little is known regarding the various factors of the problem: but nothing in the little that is known and in the much more that may be fairly inferred should be regarded as discountenancing the theory of upgrowing reefs on subsiding foundations, essentially as Darwin supposed. His primary theory of coral reefs holds good, although his supplementary theory of broad ocean-floor subsidence needs modification.

[1] Guppy, H. B., *Scot. Geogr. Mag.*, **14**, 1888, (121–137); see p. 135, 136.

[2] Hickson, S. J., *A naturalist in Celebes*, London, 1889; see p. 42.

[3] Murray, J., *Proc. Roy. Soc. Edinb.*, **10**, 1880, (505–518); see p. 516.

[4] Geikie, Sir A., *The ancient volcanoes of Great Britain*, London, 1897; see vol. 2, p. 470.

[5] Branner, J. C., *Amer. J. Sci.*, **16**, 1903, (301–316); see p. 301–303.

[6] Molengraaf, G. A. F., *Proc. k. Akad. Wet. Amsterdam*, **19**, 1916, (610–627); see p. 619–620.

[7] This statement depends on the fact, certified by chemists, that the withdrawal of limestone from solutions in water diminishes the water volume by only a small portion of the volume of the withdrawn limestone.

ON THE DEFORMATION OF AN N-CELL

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I propose to prove that *any (1 — 1) continuous transformation of an n-cell and its boundary into themselves, which leaves all points of the boundary invariant, is a deformation.*

For the purposes of this proof the n-cell may be taken to be the interior of an n-dimensional cube. A deformation is a $(1 - 1)$ continuous transformation F_1 which is a member (corresponding to $x = 1$) of a one-parameter continuous family of $(1 - 1)$ continuous transformations F_x $(0 \leqq x \leqq 1)$ such that F_0 is the identity. It is understood that each F_x is a transformation of the n-dimensional cube into a set of points of an n-dimensional Euclidean space in which the n-dimensional cube is situated.

The theorem is easy in the one-dimensional case. It has been proved in the two-dimensional case by H. Tietze (*Palermo, Rend., Circ. Mat.*, **38,** 1914, p. 247) and more simply, by H. L. Smith (in an article soon to appear in the Annals of Mathematics). No proof has been published so far as I am aware, for the higher cases.

The proofs by Tietze and Smith establish a stronger theorem than that stated above, for they show the existence of a family of transformations F_x each of which carries the square into itself and leaves all points of the boundary invariant. This restriction on the transformations F_x, that each of them shall carry the square into itself, is not needed in some of the important applications of the theorem; and without this restriction the theorem can be proved very easily.

The proof below is stated for the two-dimensional, but applies without change to the n-dimensional, case.

Let S_1 be a square, $ABCD$, whose sides are of length unity, it being understood that a square, unlike a cell, includes its boundary. Let T_2 be a translation parallel to the side AB which carries the side AD into the side BC, T_3 a translation parallel to the side BC which carries the side AB into the side DC, and T_4 the resultant of T_2 and T_3. Let S_2, S_3, S_4 be the squares into which S_1 is carried by T_2, T_3, T_4 respectively. Thus S_1, S_2, S_3, and S_4 together constitute a square whose sides are of length 2.

Let F_1 be a $(1 - 1)$ continuous transformation of S_1 into itself which leaves all points of the boundary of S_1 invariant. The transformation T_2F_1 (the resultant of F_1 followed by T_2) carries S_1 into S_2. I shall first show that T_2F_1 is a deformation and it then follows easily that F_1 is also a deformation.

The rectangle composed of S_1 and S_2 can be carried into the rectangle composed of S_3 and S_4 by a transformation Λ which for points of S_1, is the same as T_3 and for points of S_2, is the same as $T_4F_1^{-1}T_2^{-1}$. Since T_3 and $T_4F_1^{-1}T_2^{-1}$ have the same effect on the common points of the boundaries of S_1 and S_2, the transformation Λ is uniquely defined, $(1 - 1)$, and continuous.

The transformation $\Lambda.T_2F_1.\Lambda^{-1}$, as applied to S_3 is the same as $T_4F_1^{-1}T_2^{-1}T_2F_1T_3^{-1}$, which is the translation $T_4T_3^{-1} = T_2$, carrying S_3 into S_4; denote this translation T_2 by T_1. Let T_x $(0 \leqq x \leqq 1)$ denote the translation carrying S_3 a distance x in the direction of translation of T_1.

The existence of the family of translations T_x $(0 \leqq x \leqq 1)$ shows that T_1 is a deformation. But since $\Lambda.T_2F_1.\Lambda^{-1} = T_1$, $\Lambda^{-1}T_1\Lambda = T_2F_1$. Hence the existence of the set of transformations $\Lambda^{-1}T_x\Lambda$ $(0 \leqq x \leqq 1)$ shows that T_2F_1 is a deformation.

Any one of the transformations $\Lambda^{-1}T_x\Lambda$ effects a translation on the three sides, AB, CD, DA of S_1 and carries the side BC into the curve to which T_2F_1 carries the linear segment in which the square is met by a line parallel to AD at a distance x from AD. Since if T_x is the translation which has the same effect as $\Lambda^{-1}T_x\Lambda$ on the point A, the transformation $T_x^{-1}\Lambda^{-1}T_x\Lambda$ leaves all points of the three edges AB, CD, DA of S_1 invariant. Denote $T_x^{-1}\Lambda^{-1}T_x\Lambda$ by F_x. The set of transformations F_x $(0 \leqq x \leqq 1)$ is obviously a continuous one-parameter family of $(1 - 1)$ continuous transformations; F_0 is the identity; and F_1 the given transformation already denoted by F_1. Hence F_1 is a deformation.

The last paragraph can be replaced by the observation that since the product of two deformations is a deformation, F_1, which is the product of T_2^{-1} and T_2F_1, must be a deformation. It seems worth while, however, to indicate, as has been done, something of the nature of the family of transformations F_x which the process sets up.

A THEOREM ON SERIES OF ORTHOGONAL FUNCTIONS WITH AN APPLICATION TO STURM-LIOUVILLE SERIES

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1. *The Theorem.*—An infinite set of continuous functions $u_1(x)$, $u_2(x)$, . . . is *closed* on the interval $0 \leqq x \leqq 1$ if there exists no continuous function $f(x)$ not identically zero for which $\int_0^1 f(x)\, u_n(x)\, dx$ vanishes for all n; the set is *normalized* if $\int_0^1 u_n^2(x)\, dx = 1$ for all n; it is *orthogonal* if $\int_0^1 u_m(x)\, u_n(x) dx = 0$ for $m \neq n$. Most of the series of mathematical physics are linear in closed normalized orthogonal sets of functions.

THEOREM. *If $u_1(x), u_2(x)$, . . . form a closed normalized orthogonal set of functions, and if $\bar{u}_1(x)$, $\bar{u}_2(x)$, . . . form a second normalized orthogonal set such that*

$$\sum_{n=1}^{\infty} (u_n(x) - \bar{u}_n(x))\, u_n(y) \qquad (0 \leqq x,y \leqq 1)$$

converges to a function $H(x, y)$ less than 1 in numerical magnitude in such wise that the series multiplied through by an arbitrary continuous function $f(x)$ can be integrated term by term as to x and yields a uniformly convergent series, then the set $\bar{u}_1(x)$, $\bar{u}_2(x)$, . . . is closed also.

Proof. If the set $\bar{u}_1(x)$, $\bar{u}_2(x)$, . . . is not closed there exists an f not identically zero such that $\int_0^1 f(x)\, \bar{u}_n(x)\, dx$ vanishes for all n. In